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Corporate liquidity, investment and financial constraints: Implications from a multi-period model

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Abstract

In single period models, financially constrained firms invest more in response to increases in their net worth or interest rate cuts. We examine whether or not these results necessarily hold in a multi-period setting. We present a multi-period version of the Holmstrom and Tirole moral hazard model and show that the *probability* of investment (or the hurdle rate for investment) in the first period of a two-period model is non-monotonic in the level of liquid balances [Holmstrom, B., Tirole, J., 1997. Financial intermediation, loanable funds, and the real sector. Quart. J. Econ. 112 (3), 663–691. August; Holmstrom, B., Tirole, J., 1998. Private and public supply of liquidity. J. Polit. Economy 106 (1), 1–40. February; Holmstrom, B., Tirole, J., 2000. Liquidity and risk management. J. Money, Credit, Banking 32 (3), 295–319. August]. When a risk-free interest rate is introduced in the model, we show that a lower interest rate (or a downward shift or the yield curve) can lead to *less* current investment due to the interaction of future financial constraints and discounting of cash flows. Our results have implications for the effect of monetary policy on investment by financially constrained firms. They also address several recent empirical debates, such as the relationship between liquidity and the cash-flow sensitivity of investment, and whether or not accumulation of cash balances by Japanese firms can be consistent with the existence of financial constraints affecting investment.

JEL classification: E0; E3; E5; G3; G32

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1. Introduction

Why does more balance-sheet liquidity sometimes fail to stimulate investment by financially constrained firms? Why does investment not respond at times to lower interest rates? A large literature has developed based on the idea that an increase in liquidity or net worth has a positive impact on the investment levels of financially constrained firms. In this paper, we argue that in a multi-period context where firms have the option of allocating their liquidity *intertemporally*, such a presumption is not valid in general. If firms have more liquid balances and these liquid balances can be carried into future periods, then firms are less likely to be financially constrained in the future as well. Since the risk of not being able to take projects in the future—should they pass up on projects today—is lower, firms may become more conservative in their project choice *today* as their liquidity position improves. In other words, investment may *decline* today (equivalently, the hurdle rate for projects increase) as firms' liquid balances increase.

Such a perverse relationship between liquidity and investment has important policy implications. For example, the result suggests that the ability of monetary policy to stimulate firm investment and hence aggregate demand through the balance sheet channel may be may be much more limited in scope than previously thought.[2] In recessions, firms tend to accumulate cash balances as investment opportunities become less attractive. Our results show that since they are also likely to be more conservative in terms of current investment when they have larger cash balances, there is an additional channel through which investment may be affected in a downturn.

When interest rates change, we show that the interaction of financial constraints and the expected cost of borrowing in the present and the future can also have perverse effects on firms' incentives to invest. In particular, if there is a parallel downward shift in the yield curve (i.e., an equal decrease in both the first and second-period interest rates), investment in the first period decreases. The lower *current* borrowing rate increases the attractiveness of current projects and encourages current investment. However, the lower expected *future* borrowing cost also makes future projects more attractive than before. If the projects are risky, financial constraints play a role in determining the choice between current and future projects. Since failure of the current projects would imply that financially constrained firms are unable to undertake the now more attractive future projects, lower future borrowing costs tend to favor postponement of current projects. Therefore, for a positive impact on investment to occur, the yield curve has to become more positively sloped as well. This result is consistent with Estrella and Mishkin's (1998) well known result that an inverted yield curve is a precursor of recessions. It is also worthwhile to think of this result in the context of recent Japanese monetary policy. Japan has responded to deflationary conditions by setting short-term money market interest rates to zero while simultaneously engaging in quantitative easing in which the central bank has sharply increased the base money supply through orthodox and unorthodox means. One implication and potential goal of

[2] The balance sheet channel refers to the impact of monetary policy on firms' balance sheets, such as their liquid balances or net worth. For example, lower interest rates will reduce firms' short term interest expenses, thereby improving their net income. This is supposed to have a positive impact on the investment levels of financially constrained firms. See Fazzari et al. (1988), Bernanke and Gertler (1989, 1995), Gertler (1992), and Bernanke et al. (1996, 1998), for expositions of how financial market imperfections affect the impact of monetary policy on firms' investment decisions and real sector activity.

quantitative easing is an increase in expectations of future inflation (see Krugman, 1998; Eggertsson and Woodford, 2003, 2004). Given the current deflation, a high level of future inflation may imply that real interest rates are lower in the long-term than in the short-term, despite the fact that the nominal yield curve is upward sloping. Under these conditions, financially constrained firms have even more reason to postpone investment. The steady increase in the ratio of cash plus deposits to capital stock of small firms in Japan since the mid-nineties is consistent with this argument.

The perverse relation between liquidity and investment incentives also addresses a large body of empirical literature. For example, papers studying the relationship between firms' cash flows (or change in liquid balances) and investment find mixed results regarding whether or not more financially constrained firms show higher or lower sensitivity of investment to cash flows. Our results suggest why relating the degree of liquidity constraints faced by firms to differences in cash-flow sensitivities could be problematic. The same firm could exhibit a wide variation in the cash-flow sensitivity of investment depending on the level of its liquid balances; in fact, a firm's cash-flow sensitivity is non-monotonic in the level of liquid balances.[3] Our results also address an ongoing debate concerning whether or not financial constraints—especially tighter bank lending—explain the prolonged slump in investment activity by Japanese firms. Hayashi and Prescott (2002) point to the huge cash buildup by Japanese firms since the early 1990s to argue that financial constraints could not have explained the failure of investment to pick up in Japan. Our analysis shows that, first, a cash build up in anticipation of future financial constraints itself could further exacerbate a slump, as firms become more conservative about investing today; moreover, the huge cash build up itself, in part, might have been in anticipation of future financial constraints.

We demonstrate our result in the context of a model that represents a multi-period extension of the well-known moral hazard model of Holmstrom and Tirole (1997, 1998, 2000). Each period, a firm draws a project at random. The project generates risky cash flows one period later. In this model, firms are financially constrained in the sense that given insufficient stock of liquid balances at the beginning of each period, they cannot invest in all positive net present value projects because of a moral hazard problem vis-à-vis the firm and external investors. However, in our multi-period version of the model, if firms have more liquid balances today and these liquid balances can be carried into future periods, then firms are less likely to be financially constrained in the future. Thus, the higher liquidity lowers the risk that the firm is not able to take projects in the future, should it pass up on projects today. As a result, the firm can become more conservative in its choice of projects *today* if its liquidity position improves, and the probability of investment can decline in the level of its liquid balances. We show that in a two-period extension of the Holmstrom–Tirole model with exogenously fixed scale of investment, the probability of investment in the first period is non-monotonic in the level of liquid balances under very general conditions. In particular, for low levels of liquid balances, the probability of investment increases in the level of liquid balances; it decreases in the level of liquid balances for intermediate levels, before increasing again for high levels of liquid balances.[4]

Several other papers have addressed the issue of the non-monotonic relation between changes in liquidity and investment, although none explore the intertemporal issues that we stress in

[3] The fact that the liquidity position of a firm is itself endogenous further complicates the comparison.

[4] Such a result also obtains when the model is extended to more than two periods. See Section 5.

this paper.[5] Kaplan and Zingales (1997) work in terms of a reduced-form model in which the cost function for external finance is increasing and convex, and show that the way in which the degree of financial constraints (a parameter in the cost function for external finance) affects the sensitivity of investment to changes in liquidity is, in general, ambiguous. Almeida and Campello (2002) use the framework of Hart and Moore (1994) to argue that an increase in liquidity will "relax" the constraints less for firms with greater scope for diverting the liquidation cash flows from investment (the "more constrained" firms). Since external lenders will recover a smaller proportion of these cash flows as they step up lending, more constrained firms will exhibit smaller sensitivity of investment to changes in liquidity.

The paper that is closest to ours is Boyle and Guthrie (2003). In that paper, an entrepreneur can invest at any moment in a project whose value evolves according to a geometric Brownian motion. Financial constraints are exogenous: the firm cannot invest more than its available stock of cash (which also follows a stochastic process) and a fraction of the project value. Boyle and Guthrie (2003) show by way of numerical analysis that, as a result of uncertainty about the extent of the financial constraint, the firm can exercise its option to invest too soon. However, an increase in cash balances—while relaxing the financial constraint—can also make waiting less risky, and thus cause late exercise of the investment option. This model shares several features with our first model—in particular, the exogenous scale of investment. Unlike Boyle and Guthrie (2003), however, we explicitly model the firm's financing decision with endogenous financial constraints.[6]

The rest of the paper is organized as follows. Section 2 outlines the model and presents the results on the relationship between liquid balances and the hurdle rate. Section 3 offers a discussion of the model's relationship with other multi-period models of investment, and the relevance of the main results. Section 4 generalizes the basic model when the risk free interest is positive and explores the relationship between interest rate changes and investment. Section 5 considers extensions to an arbitrary but finite number of periods, and Section 6 concludes.

2. The model

We consider a two period model with three dates, $t = 1, 2, 3$. At time $t = 1$, a risk-neutral entrepreneur (henceforth denoted by E) with a cash endowment of c can take a project. The project requires an investment of 1 unit of money and generates at $t = 2$ (i.e., the start of the second period), a cash flow of $a_1 > 0$ with a probability p and 0 with probability $1 - p$. We refer to a_1 as the "state of the project" in period 1. We will assume throughout that at the time of investment, the state of the project is common knowledge.

At the beginning of the second period ($t = 2$), E has an opportunity to take a new project. This new project also requires 1 unit of money, and succeeds with probability p.[7] In the event of success, the project pays a_2 at the end of the second period ($t = 3$), and zero in the event of failure. The state of the project at $t = 2$ is denoted a_2. We assume that a_2 is distributed on $[\underline{a}, \bar{a}]$ with a continuous distribution function $F(\cdot)$ and density function $f(\cdot)$. The realization of the

[5] The significance of intertemporal issues, however, is often raised in the literature. See Fazzari et al. (2000, p. 701) and Gilchrist and Himmelberg (1998).

[6] Other papers that also look at intertemporal tradeoffs between investing today and investing in the future are Whited (1992) and Whited and Wu (2006). However, these models do not consider the intertemporal allocation of liquidity, as we do.

[7] None of our results change if we assume a different probability of success for the second period project.

state at $t = 2$ is assumed to be common knowledge. Prior to $t = 2$, only the distribution $F(\cdot)$ is known to E as well as the market. Investors are assumed to be risk-neutral.

Let a^* be the state of a project such that $pa^* = 1$. Clearly investment is profitable if and only if the state of the project exceeds a^*.[8] Throughout, we assume that $\underline{a} < a^* < \bar{a}$.

In each of these periods, E also has the option of investing 1 unit of money in an alternative project. The alternative project does not generate any verifiable cash flow but gives E a private benefit of B where $B \leqslant 1$. We will refer to this alternative project as the "unproductive project" and E will be said to invest "unproductively" if he chooses this project. Existence of this alternative project is the source of "moral hazard" and the resulting imperfection in the credit market.

If E's cash endowment in any period is less than 1, he will need to raise external funds to finance the project. For most of the paper, we restrict our analysis to simple debt contracts in which period t debt holders only have claims to cash flows realized in period $t + 1$.[9] This implies in particular that debt holders financing the project at $t = 1$ only have a claim on the cash flows at $t = 2$, i.e. from the project undertaken at $t = 1$.[10]

To understand E's investment decision in this dynamic setting, we first consider the last period. Let $x < 1$ be the amount of cash that E has at his disposal at the beginning of the second period (after meeting his first period debt obligations, if any), and assume that state a_2 is realized.

For E to invest in this state, he must be able to raise the additional amount of $1 - x$ by means of a one-period debt contract. For lenders to be willing to lend, at the face value of debt d, E must invest productively (otherwise lenders get nothing), and the lenders must break even. Hence, we must have

$$p[a_2 - d] \geqslant B \tag{1}$$

where $pd = (1 - x)$.

At this point, it is useful to point out that the moral hazard that we are modelling can be given alternative interpretations. For example, suppose that E can choose between two levels of effort, high and low, and B represents E's utility from shirking. Choosing the low level of effort (shirking) causes the project to fail with certainty, while choosing the high level of effort causes it to succeed with probability p. This is, in fact, the Holmstrom and Tirole (1997, 1998) model.

A less obvious, but important, interpretation is that of the standard risk-shifting behavior. Here, we can think of E as a manager who is managing the firm in the interest of its owners. The conflict of interest associated with risk-shifting behavior is between owners and lenders. Suppose that E in any period can choose an alternative project that pays an amount A with probability q and zero with probability $1 - q$. Let $B = q(A - d)$, where d is the promised payment to debt holders. Make q very small and A very large so that $p(a_2 - d) < q(A - d) = B$ holds. If the alternative project is chosen, debt holders (almost) never get paid (since q is very small), but equity holders prefer this project.

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[8] For now, we assume that the risk-free interest rate is zero. In Section 4, we relax this assumption to examine the effect of interest rate changes on E's incentive to invest.

[9] In Section 5, we discuss how our results are affected if E is allowed to finance his funding shortfall at $t = 1$ by issuing equity shares.

[10] In an earlier version of the paper, we considered debt contracts that allow period 1 debt holders to be paid, in the event of default, cash flows from the period 2 project as well. It was shown that for a range of cash holdings in period one, the debt contract analyzed here turns out to be optimal in a larger class of debt contracts, and our main results continue to hold.

It is clear from (1) that if E is able to invest in state a_2, then he is able to invest in all states $a' > a_2$. Moreover, given x, the lowest state for which (1) is satisfied is given by $(B+(1-x))/p$. Since E will never invest in a state $a_2 < a^*$ in period 2, it follows that E with cash holding x will invest if and only if $a_2 \geqslant a_x$ where a_x is defined as

$$a_x = \max\left\{\frac{B+(1-x)}{p}, a^*\right\}. \tag{2}$$

Observe that a_x is (weakly) decreasing in x and thus if E with cash holding x can invest in state a, he can also invest in that state with a higher cash holding. Since $B > 0$, it is clear that a_0—the lowest state in which E is able to invest with zero cash holding—satisfies $a_0 > a^*$. Thus, with a zero cash holding in period 2, E will not be able to invest in all productive states. It is in this sense that a firm with no or limited cash in the second period is financially constrained.

The ex ante (i.e. prior to the draw of the second project at $t = 2$) payoff to E from having x units of cash in period 2 is then given by

$$\begin{aligned} P(x) &= \int_{a_x}^{\bar{a}} p\left[a - \frac{(1-x)}{p}\right] \mathrm{d}F + F(a_x)x \\ &= \int_{a_x}^{\bar{a}} [pa-1]\,\mathrm{d}F(a) + x. \end{aligned} \tag{3}$$

Possible shapes of the payoff function are shown in Figs. 1 and 2. In these figures, the distribution function $F(a)$ is assumed to be uniform. In Fig. 1, the payoff function has an initial linear stretch and passes through the origin. This reflects the fact that for low enough x, E may not be able to invest at all. In Fig. 2, the payoff function has a positive intercept on the vertical axis. This corresponds to the case in which even with zero cash, E can invest is some positive NPV projects.

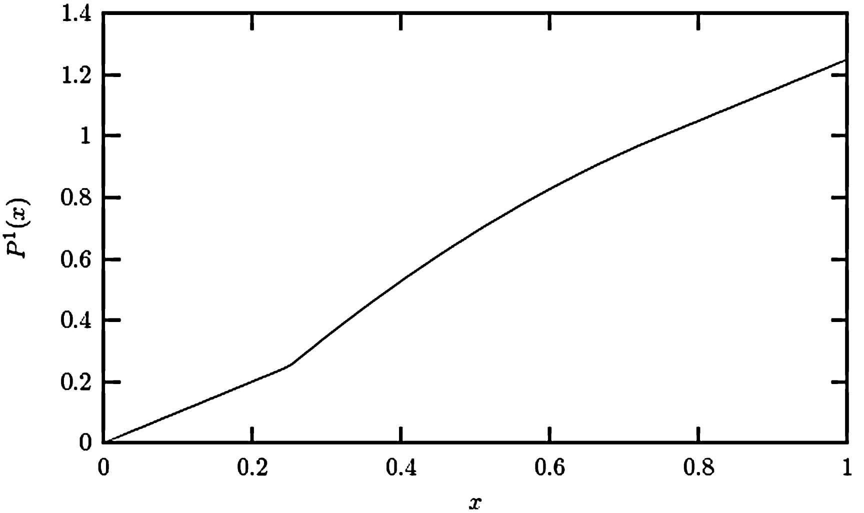

Fig. 1. Single-period payoff P^1 as a function of current cash balance x for the case $P^1(0) = 0$: $p = 1/2, \bar{a} = 3, B = 3/4$.

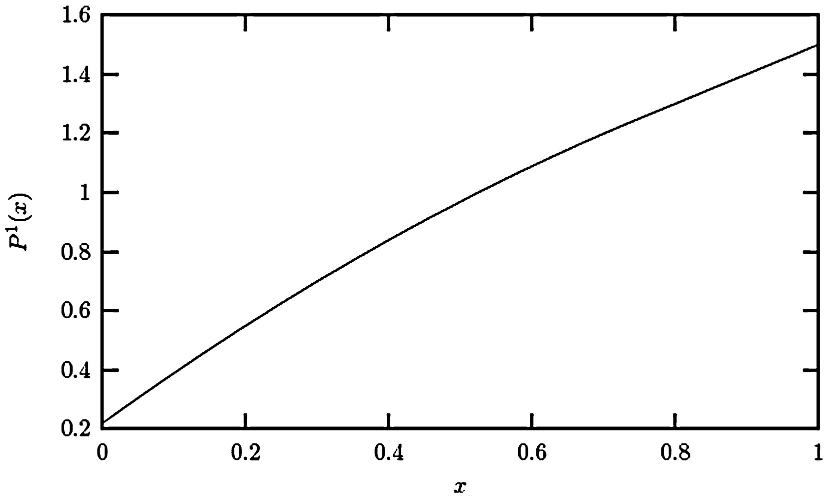

Fig. 2. Single-period payoff P^1 as a function of current cash balance x for the case $P^1(0) > 0$: $p = 1/2$, $\bar{a} = 4$, $B = 3/4$.

We now turn to the $t = 1$ investment choice of E. Suppose that state a_1 is realized. If E raises $1-c$ by issuing debt with face value D and invests productively, then, the debt holders get back D only when a_1 is the outcome; they get back zero otherwise. Thus, for the lenders in period 1 to break even, we must have $pD = (1 - c)$.[11] Furthermore, it must be in the interest of E to invest productively. This requires that the following Incentive Compatibility (*IC*) constraint must hold:

$$pP(a_1 - D) + (1 - p)P(0) \geqslant B + P(0). \qquad (IC) \tag{4}$$

Moreover, E can always decide not to invest and carry over the cash c to next period. Thus, we must have the following Individual Rationality (*IR*) condition:

$$pP(a_1 - D) + (1 - p)P(0) \geqslant P(c). \qquad (IR) \tag{5}$$

Hence, E with cash holding c invests in state a_1 in period 1 if and only if *IC* and *IR* constraints are both satisfied. Moreover, if these constraints are satisfied for some state a_1, then they will be satisfied for all $a_1' > a_1$. Let a^c be the lowest state in which E with cash holding c invests in period 1.[12]

2.1. Liquid balances and the hurdle rate

We now discuss how the Incentive Compatibility and Individual Rationality constraints come into play and affect E's investment behavior at $t = 1$. At an intuitive level, one expects that if E has to borrow more (i.e., c is small), then his payoff after paying back the debt holders would be

[11] We assume that if E borrows, and carries over any cash into the second period, then debt holders can claim that cash in the event of default. This assumption is justified in situations when the success or failure of the project is not verifiable and thus an incentive compatible debt contract must allow debt holders to have a prior claim on E's assets in the event of default. Our results however do not depend on this assumption. In a working paper (Dasgupta and Sengupta, 2002), we allow for the possibility that the entrepreneur and lenders at $t = 1$ agree that the former can carry over and set aside an amount at $t = 2$ even in the event of default. All our results generalize.

[12] We use superscripts to denote marginal states at $t = 1$ and subscripts to denote marginal states at $t = 2$.

small, and thus E would have a greater incentive to undertake the unproductive investment and enjoy the private benefit. Hence for low values of c, one expects the (*IC*) constraint to bind at the marginal state a^c. On the other hand, for a sufficiently high value of c, the (*IR*) constraint should bind. We show below that there is a unique level of cash balance, c^0, defined by

$$P\left(c^0\right) = P(0) + B \tag{6}$$

such that the *IC* (*IR*) constraint binds if $c < c^0$ ($c > c^0$).

Claim 1. c^0 *exists, is unique and satisfies* $0 < c^0 < B$.

Proof. Please see Appendix A. □

Proposition 1. (i) *For all $c < c^0$, the IC constraint binds at the marginal state a^c and the IR constraint does not bind.*

(ii) *For all $c > c^0$, the IR constraint binds at the marginal state a^c and the IC constraint does not bind.*

Proof. Please see Appendix A. □

To understand Proposition 1, it is helpful to interpret $P(c)$ as E's ex ante ($t = 0$) payoff from the "average" project next period, where the average is with respect to all states of nature—including those in which, due to the nature of the financial constraint discussed previously, E is not able to invest at all (or, equivalently, invest in a risk free zero NPV project). When c is low, E is only able to invest in very good projects in the first period. This is because unless his payoff net of repaying the debt holders is sufficiently high, he prefers diverting one dollar to the unproductive project and enjoy the private benefit. Thus, the marginal project for which the *IC* constraint holds is of high productivity; moreover, since c is small, the "average" project next period does not provide a high payoff since he will be unable to invest in many positive NPV projects. Thus, the marginal project for which the *IC* constraint holds today is better than the average project next period, implying that the *IC* constraint binds and the *IR* constraint is slack. For high values of c, he is able to invest in projects of lower NPV; thus, the marginal project for which the *IC* constraint holds is not as good as the average project next period. As a result, the *IC* constraint must now be slack, and the *IR* constraint must bind.[13]

An important issue related to financial constraints is how the firm's hurdle rate for today's projects is affected by its current liquidity. Notice that if the state of the marginal project a^c is decreasing in c, then the firm will accept a project that it would have rejected at a lower level of c, which implies that the hurdle rate decreases in c. Is the firm's hurdle rate necessarily decreasing in its aggregate cash holding c? We address this now.

[13] In our framework, projects differ only with respect to the end-of-period payoff in the non-default state. Projects, however, could differ in other respects, such as the probability of success, or in terms of whether they pay off earlier or later. Thakor (1990) models the choice between projects that pay off earlier and those that pay off later, and shows that a financially constrained firm might prefer a lower NPV project if it pays off earlier. Our result that the *IR* constraint is slack for low levels of c implies that if the state is slightly below the marginal state, the firm does not invest even though the expected payoff would have exceeded that from carrying the cash to the next period. To the extent that investing in a faster payback project is similar to generating future liquidity faster, this is very reminiscent of Thakor's (1990) result.

Proposition 2. *For any $c < c^0$, a^c is decreasing in c.*

Proof. For $c < c^0$, we know from Proposition 1 (part (i)) that the *IC* constraint binds in the marginal state. Thus for any $c < c^0$, we have

$$p\big[P\big(a^c - (1-c)/p\big) - P(0)\big] = B.$$

If c increases $(1-c)/p$ falls and thus a^c must fall as well. □

At $c = c^0$, Proposition 1 indicates that the binding constraint for the marginal state switches from the *IC* constraint to the *IR* constraint. This has immediate implications for the sensitivity of the marginal project to changes in liquid balances as shown in the following proposition.

Proposition 3. *The slope of the $a^c(c)$ function changes* discontinuously *at $c = c^0$, and $\lim_{c \uparrow c^0} \frac{\mathrm{d}a^c}{\mathrm{d}c} < \lim_{c \downarrow c^0} \frac{\mathrm{d}a^c}{\mathrm{d}c}$.*

Proof. Please see Appendix A. □

We saw in Proposition 2 that the slope of the $a^c(c)$ function is negative for $c < c^0$. Proposition 3 indicates that this slope becomes less negative—and possibly positive—for $c > c^0$. In other words, the impact of a change in liquidity on investment could be muted if firms' current cash balances exceed c^0. In fact, it can be shown that the slope of the a^c function is more steep for $c < c^0$ than for $c > c^0$, implying that the response of the probability of investment or the hurdle rate to changes in liquidity is generally lower for firms with liquid balances above c^0 than those with liquid balances below c^0. We postpone a more complete discussion of this and related issues until later. For now, we show that, in the two-period setting, we can in fact get a much stronger result regarding the way a change in c affects the marginal project (equivalently, E's hurdle rate for projects), namely, the hurdle rate *increases* for a range of cash holding beyond c^0.

Proposition 4. *There exists c^*, $c^0 \leqslant c^* < B$ such that a^c is strictly increasing in the interval (c^*, B).*

In an earlier version of the paper, it was shown that if $P(\cdot)$ is a strictly concave function over the range of cash holding (c^0, B), then a^c is increasing over the entire domain (c^0, B). The present proposition does not require any such curvature assumption on $P(\cdot)$; the result accordingly is weaker. In Proposition 4′, we are able to prove a stronger version of this non-monotonicity result for the entire range of cash holdings (c^0, B) when project the success probability p is not too high.

Proof. From Proposition 1 (ii), we know that *IR* constraint must bind for all $c > c^0$ at the marginal state a^c. Let $y(c) = a^c - \frac{(1-c)}{p}$. We then have

$$pP\big(y(c)\big) + (1-p)P(0) = P(c).$$

Totally differentiating the above identity with respect to c, one gets

$$pP'\big(y(c)\big)\left[\frac{\mathrm{d}a^c}{\mathrm{d}c} + \frac{1}{p}\right] = P'(c)$$

or

$$pP'\big(y(c)\big)\frac{\mathrm{d}a^c}{\mathrm{d}c} = P'(c) - P'\big(y(c)\big). \tag{7}$$

Recall that

$$P(c) = c + \int_{a_c}^{\bar{a}} (pa_2 - 1)\,\mathrm{d}F$$

where a_c from Eq. (2) is given by $a_c = \max\{\frac{B+(1-c)}{p}, a^*\}$. Notice now that for $c > B$, $a_c = a^*$ and thus $P'(c) = 1$. On the other hand, for $c < B$, we have $a_c > a^*$ and

$$P'(c) = 1 - [pa_c - 1]f(a_c)\frac{\mathrm{d}a_c}{\mathrm{d}c}.$$

Since from Eq. (2) $\frac{\mathrm{d}a_c}{\mathrm{d}c} = -1/p$ and $pa_c - 1 = B - c > 0$, it follows that $P'(c) > 1$ for $c < B$.

From the *IR* constraint, it follows that $y(c) > c$ for $c > 0$; moreover, $y(c)$ is increasing in c. Thus, there exists $c^* < B$ such that for $c \in (c^*, B)$, $y(c)$ is strictly greater than B. Since $P'(c) > 1$ and $P'(y(c)) = 1$ for any such c, it follows from Eq. (7) that $\frac{\mathrm{d}a^c}{\mathrm{d}c} > 0$ and the result is proved. □

Remark. Notice that the slope of a^c as a function of c becomes flat for $c \in (B, 1)$, since both $P'(c)$ and $P'(y(c))$ equal one. For $c > 1$, a^c will decrease in c and as c approaches the value of $1 + B$, a^c converges to $a^* = 1/p$.

The intuition for the non-monotonicity result is as follows. At low levels of c, E is unable to take some even very good projects because of the severity of the moral hazard problem. The marginal project gives a higher payoff than the "average project" that he will be able to invest in if he carries the c to the next period. Although there is a positive probability that the project taken today may fail and E will then only obtain a payoff of $P(0)$ next period, when c is small, the alternative option of not investing today is also not sufficiently attractive. In other words, the *IC* constraint binds but the *IR* constraint does not. Any increase in c allows him to undertake additional projects today that are of somewhat lower quality, but still better than the average project he can invest in next period. Thus, for low values of c, the state of the marginal project must decrease in c. As c increases, the benefit from not investing today increases, since he faces a lower risk of being constrained and not being able to invest at all next period. The cost associated with investing today, namely the probability of going to the next period with zero cash remains the same at $(1 - p)$. Therefore, for c sufficiently high, the *IR* constraint binds. Notice that an increase in c increases the payoff both from investing today (since he needs to borrow less and has more cash next period conditional on success) as well as from investing next period (since he can invest in more projects). When c is sufficiently close to 1, however, the former benefit increases less rapidly, since the firm is already able to invest in all positive NPV projects conditional on success. Since, ceteris paribus, investing later becomes relatively more attractive for higher c, the marginal project today has to be even better.

The intuition behind this result suggests that E will become even more conservative about investing today when the success probability of a project p decreases. Consistent with this intuition, for a sufficiently low value of p, we are able to get a stronger characterization:

Proposition 4′. *Suppose $p \leqslant 1/2$, then a^c is strictly increasing in the interval (c^0, B).*

Proof. Please see Appendix A. □

3. Discussion

3.1. Relationship with the real options literature

Some of our results in this section have the flavor of results from the real options literature. In this literature, early adoption of a project precludes late adoption, and as a consequence, some positive NPV projects are not taken. For example, McDonald and Siegel (1986) consider a model in which a firm has a perpetual right to a project that involves uncertain future value and irreversible investment costs. They find that the optimal investment policy involves taking the project only when the NPV exceeds a positive threshold.

In the present set up, it is possible to show that the cut-off for the first period is indeed above a^* under the added restriction that the $P(\cdot)$ is strictly concave for $c < B$.[14] Thus in the first period, E passes up on some positive NPV projects. Such a result however will obtain even in other multi-period models without moral hazard. It is thus worthwhile to examine what moral hazard is buying us here, and the connection of our results to the real options literature.

First, notice that the presence of moral hazard in the second period is crucial for our 'non-monotonicity' result. Without any such friction in the second period, the model will collapse to a one period model in which the *IC* constraint will always bind at the marginal state a^c for any c. Consequently, a^c must be decreasing in c, i.e., current investment can not decrease with an increase in the present cash holding.

On the other hand, if one has a set up with moral hazard only in the second period, then it is easy to check from Eqs. (4) and (5) that the *IR* constraint will bind for all $c \geqslant 0$. Thus, at $c = 0$, one will have from (5) $P(a^c - 1/p) = P(0)$ or that $a^c = a^*$. Thus, without moral hazard in the first period, a firm with no cash will still invest in all positive NPV projects. While Proposition 4 and Proposition 4′ remain valid in this setting, the strength of our approach with moral hazard in both periods is that it provides us with theoretical predictions of both types of investment behavior. The prediction that a non-monotonic response of investment to the initial cash holding is expected only when the cash holding exceeds a critical level seems to us to be more in line with the stylized facts.

Interestingly, uncertainty about the second period project is necessary for our non-monotonicity result. In this respect, there is a similarity with the real options literature, where also uncertainty about the future state of projects is crucial.[15] However, the similarity ends there. In Appendix B, we show that greater uncertainly in the second period can lead to greater period one investment and lower firm value, an outcome that can not obtain under a real options framework. The intuition for this result is as follows. The moral hazard constraint precludes investment in the low states in the second period. A mean-preserving spread in the distribution of the state of the second-period project, for some levels of cash holdings at the beginning of the second period, increases the probability that the firm is not able to invest in the low states. This lowers $P(c)$, the payoff from not investing in the first period and carrying c to the second period. If the *IR* constraint binds, then the marginal project in which the firm invests in the first period must also be lower. Moreover, since the payoff to the firm is the higher of what it can get by investing today

[14] If $P(0) > 0$, then $P''(c) < 0$ for $c \in [0, B)$ if, for instance, $F(\cdot)$ is uniform. More generally, concavity of $P(\cdot)$ is analogous to the concavity of the payoff function in Froot et al. (1993), and is consistent with E having a hedging motive.

[15] If the second period project is known with certainty, in the present set up one still gets a strictly positive NPV cut-off ($a^c > a^*$). a^c is decreasing for $c < c^0$. However, beyond c^0, a^c remains constant.

and carrying the cash to the next period, and the latter is lower, it follows immediately that the overall two-period payoff must also be lower.

3.2. The relevance of the non-monotonicity result

A large literature finds that certain types of firms (the so-called "less financially constrained firms", such as larger firms, firms with business group affiliations, or those with debt and commercial paper ratings) show less sensitivity of investment to changes in internal funds or liquidity than more financially constrained firms (see Hubbard, 1998 for a survey). This issue is important because it shows the relevance of information problems firms face vis-à-vis external providers of finance for their financing and investment decisions. Thus, one implication of the existence of such problems is that financial development will affect the growth rates of firms by reducing the extent of these information problems (Rajan and Zingales, 1998).

Recent studies, however, have found some puzzling empirical evidence about the cash flow sensitivity of investment. For example, Kaplan and Zingales (1997) and Cleary (1999, 2004) find that the sensitivity of investment to cash flows is higher for firms with better liquidity positions at the beginning of a particular year than those with worse liquidity positions. They suggest that these results are inconsistent with the idea that financially more constrained firms will exhibit higher cash flow sensitivity of investment. Since cash inflows add to the stock of liquid balances, the impact of cash flow on investment will be analogous to that of a change of c on investment. Our analysis suggests that the cash-flow sensitivity of investment may well depend on the liquidity position of firms. Firms with liquidity in a neighborhood of c^0 may show a much lower sensitivity of investment to change in liquidity than firms with c much higher than c^0,[16] which is consistent with empirical evidence.

The non-monotonicity result has also major implications for the conduct of monetary policy (explored below), or how corporate cash holdings affect firms' hurdle rates for projects. If firms have poor projects in an economic downturn, they would accumulate cash balances. Our results indicate that this may have the effect of making firms even more conservative, leading to further reduction in investment, and possibly magnifying the recession.

4. The effect of monetary policy: lowering interest rates

Lowering interest rates is a standard tool of monetary policy for stimulating economic activity. In this section,[17] we show that interest rate cuts can actually *discourage* investment unless the slope of the yield curve also changes. In particular, short term interest rates may have to fall more than long term rates in order to stimulate current investment. The intuition is that lower future interest rates make future projects more attractive. Firms may therefore be reluctant to invest in risky (but positive NPV) projects today to preserve the option of investing in the future. To encourage current investment, therefore, the short term interest rate has to fall as well. In the context of our two-period model, a parallel downward shift of the yield curve does not completely ameliorate the perverse effect—the first-period rate has to fall more than the second-period rate for investment in the first period to respond positively.

[16] Recall that for $c > 1$, $a^c(c)$ is decreasing in c, and converges to a^* as c approaches $1 + B$.

[17] We are grateful to an anonymous referee who suggested that we formally look into the issues considered in this section.

Incorporation of a positive risk free interest in our earlier set up is straightforward. Let r_1 and r_2 denote, respectively, the first and second period interest rates. First consider period 2 and suppose E has $x < 1$ amount of cash at his disposal and that state a_2 is realized at $t = 2$. As before, if E decides to invest and borrows $(1 - x)$, he must invest productively and the lenders should get a net return of r. Thus

$$p[a_2 - d] \geqslant B \tag{8}$$

where $pd = (1 - x)(1 + r_2)$. B, the payoff from investing unproductively is assumed now to be less than $(1 + r_2)$ and is realized at the end of the period. It follows that E with cash holding x will invest if and only if $a_2 \geqslant a_x(r_2)$ where $a_x(r_2)$ is defined as

$$a_x(r_2) = \max\left\{\frac{B + (1 - x)(1 + r_2)}{p}, a^*(r_2)\right\} \tag{9}$$

where $a^*(r_2)$ satisfies $pa^*(r_2) - (1 + r_2) = 0$.

The ex ante payoff to E from having x units of cash in period 2 is then given by

$$P(x; r_2) = x(1 + r_2) + \int_{a_x(r_2)}^{\bar{a}} \left[pa - (1 + r_2)\right] \mathrm{d}F(\cdot). \tag{10}$$

Turning now to period 1, if E raises $(1 - c)$ to invest when state a_1 is realized, the face value of the debt D must equal $D = (1 - c)(1 + r_1)/p$. Furthermore, the (*IC*) and the (*IR*) constraints must be satisfied as well. Let $c_r = c(1 + r_1)$. Assuming that E only consumes at the end of the second period, the (*IC*) constraint is given by[18]

$$pP(a_1 - D; r_2) + (1 - p)P(0; r_2) \geqslant B + P(0; r_2) \tag{11}$$

while the (*IR*) constraint is given by

$$pP(a_1 - D; r_2) + (1 - p)P(0; r_2) \geqslant P(c_r; r_2). \tag{12}$$

As before, one can now show that there exists[19] $c^0 < B/(1 + r_1)(1 + r_2)$ such that if $c < c^0$, the (*IC*) constraint will bind at the marginal state a^c while for $c > c^0$, the *IR* constraint will bind.

We will show that the probability of investment in the first period could decrease when there is a parallel downward shift of the yield curve, i.e., $\mathrm{d}r_1 = \mathrm{d}r_2 = \mathrm{d}r < 0$.

Example. Assume that in period 2, only one state a_2 will be realized.[20] Assume (i) $\frac{p(1-p)a_2}{(1+r_2)^2} > 1$ and (ii) $pa_2 - (1 + r_2) < B$ where $B = 1$.

It is easy to check that there exists values of (a_2, p) for which both these inequalities can be satisfied for a range of r_2.[21] The implication of condition (ii) is that if E carries zero cash next period, he will not be able to invest next period and obtain a zero payoff, i.e. $P(0, r_2) = 0$.

18 If however E were to consume B at the end of the first period, the *IC* constraint needs to be modified as follows: $pP(a_1 - D; r_2) + (1 - p)P(0; r_2) \geqslant B(1 + r_2) + P(0; r_2)$. No change in our analysis is required if one were to adopt this latter convention.

19 c^0 is now defined by $P(c^0(1 + r_1), r_2) = B + P(0, r_2)$.

20 The example here can be easily extended to accommodate uncertainty about the state of the project in the second period.

21 Take $a_2 = 6$, $p = 1/4$, then both inequalities are satisfied for r_2 sufficiently small.

Condition (i) holds only if p is not too close to 1 or 0. It will become clear below why this condition is needed.

Given any c, let $c_r = c(1+r_1)$ satisfy $pa_2 - (1-c_r)(1+r_2) > 1$. It is easy to check from Eq. (9) that with c_r cash holding at $t = 2$, E will be able to invest next period. Then his total income next period is

$$P(c_r, r_2) = (c_r - 1)(1+r_2) + pa_2.$$

Given a state a^c in period 1, if E decides to invest, then his expected payoff, assuming that he invests productively in period 1, is given by

$$Y\left(a^c, r_1, r_2\right) = p\left[\left(a^c - \frac{(1-c)(1+r_1)}{p} - 1\right)(1+r_2) + pa_2\right].$$

Clearly there exists c^0 such that if $c > c^0$, then the *IR* constraint must hold at the marginal state a^c, i.e. $Y(a^c, r_1, r_2) = P(c_r, r_2)$. Expanding these expressions and canceling the common term, we get

$$pa_2 - (1+r_2) = (1+r_2)\left[pa^c - (1+r_1)\right] + p\left[pa_2 - (1+r_2)\right]. \tag{13}$$

Consider first the case of an equal change in r_1 and r_2, i.e., $\mathrm{d}r_1 = \mathrm{d}r_2 = \mathrm{d}r$. This is the case of a parallel shift of the yield curve. Differentiating both sides of (13) and rearranging, we get

$$\begin{aligned} -(1+r_2)p\frac{\mathrm{d}a^c}{\mathrm{d}r} &= (1-p) - (1+r_2) + \left(pa^c - (1+r_1)\right) \\ &= \frac{p(1-p)a_2}{1+r_2} - (1+r_2) \end{aligned} \tag{14}$$

where we have used (13) to substitute for $pa^c - (1+r_1)$.

Since $\frac{p(1-p)a_2}{(1+r_2)^2} > 1$ by assumption, we have $\frac{\mathrm{d}a^c}{\mathrm{d}r} < 0$.

Thus, a parallel downward shift of the yield curve can lead to *lower* current investment. To understand the intuition for this result, notice from Eq. (13) that a lower r_1 and a lower r_2 have opposite effects on the incentive to invest in the first period. A lower r_1, ceteris paribus, increases the right-hand side of the above expression (essentially, increases the NPV of the first period project) and therefore for the *IR* constraint to still hold, a^c must decrease. In other words, investment in the first period will be more likely, ceteris paribus. However, a decrease in r_2 raises the NPV of the second period project, which appears on both sides of (13). Since $p < 1$, the left-hand side increases more than the right-hand side, implying that a^c must rise to restore equality. The intuition here is that since the first period project can fail, a more attractive second period project will cause the firm to be more conservative in investing in period one.

The fact that a decrease in the second period rate r_2 can lead to lower investment in the current period is consistent with Estrella and Mishkin's (1998) well known result that an inverted yield curve is a predictor of recessions. What our results show is that a parallel downward shift of the yield curve aimed at stimulating investment activity may not have the desired effect—rather, the short rate has to fall well below the long rate for current investment to respond positively to interest rate cuts. In Japan, monetary authorities have set the short-term interest rates to zero while at the same time resorting to "quantitative easing" of the monetary base, which may well imply a high long term expected inflation rate (see Krugman, 1998; Eggertsson and Woodford, 2003, 2004). This has the effect of potentially making the long term real interest rate lower than the short term real rate, even though the nominal yield curve is upward sloping. Financially constrained firms are likely to postpone investment and accumulate cash under such circumstances. The huge cash build up of Japanese firms[22] is consistent with this argument.

5. Equity contracts

In this section, we assume that E can use equity to finance its investment in any period. However, given our present formulation of moral hazard, it is immediate that for the last period, the exact form of the financial contract is immaterial as long as this contract provides E with the incentive to invest productively and the investors break even. Consequently, without loss of generality, we continue to assume that the last period contract is financed using a debt contract. We assume also that the risk free interest rate is zero.

Now suppose that in period 1, E has invested productively after issuing a fraction $(1-\alpha)$ of the firm's equity. Consider period 2. If the project results in a failure, E will have zero cash holding and thus to invest, he has to raise the entire investment requirement by issuing debt with face value $1/p$ at $t=2$. For E to invest productively at the realized state a_2, incentive compatibility now requires that E's share of profit from investing must be at least B. Since E holds α proportion of the firm, the IC constraint in the second period now becomes

$$\alpha p[a_2 - 1/p] = \alpha[pa_2 - 1] \geqslant B. \tag{15}$$

Clearly, if E can invest in some state a_2, he will be able to invest in all higher states. Accordingly, the expected payoff to the firm with outstanding equity of $(1-\alpha)$ and when the first period project results in a failure is given by

$$H_0(\alpha) = \int_{a_\alpha}^{\bar{a}} [pa - 1]\, \mathrm{d}F(\cdot)$$

where $a_\alpha = \frac{B}{\alpha p} + \frac{1}{p}$.

It is interesting to note that for $\alpha < 1$, we have $a_\alpha > a_0$ (the cut-off when investment in both periods is financed with a one period debt contract and the first-period project results in a failure). Since $H_0(\alpha)$ is increasing in α, we also have $H_0(\alpha) \leqslant P(0)$, the expected payoff to E if the first period project fails and a one-period loan finances both projects. Equity contracts thus have worse second period incentives conditional upon the first-period project being a failure.

Now suppose that in period 1, the first period project results in a success. Assuming that the state a_1 for which investment was undertaken in period 1 has $a_1 > 1$, if E wants to invest in period 2, he does not need to borrow. However, since E holds only a fraction α of the firm, for low states in period 2, E will not have the incentive to invest productively. To ensure that such unproductive investment does not take place in period 2, we follow the existing literature in assuming that the outside shareholders have the power to liquidate the firm at the end of period 1.

Now given $a_2 \geqslant a^*$, and that E has 1 to invest, he will invest productively in this state if and only if $\alpha p a_2 \geqslant B$. Thus for all a_2 less than a^α where $a^\alpha = \max\{a^*, B/\alpha p\}$, the firm will be liquidated, while investment will be undertaken for all states higher than this critical value. Thus the expected payoff to the firm with outstanding equity of $(1-\alpha)$ and when the first period

[22] In Japan, small firms have steadily increased the ratio of cash plus deposits to capital stock since 1996 (see Hayashi and Prescott, 2002). Yet, this apparent improvement in their liquidity position has failed to spur investment.

project a_1 results in a success is given by

$$H_1(\alpha, a_1) = a_1 + \int_{a^\alpha}^{\bar{a}} [pa - 1]\,\mathrm{d}F(\cdot).$$

Unlike H_0, the comparison of $H_1(\alpha, a_1)$ with $P(a_1 - D)$, the payoff of the firm conditional on success of the period-one project when a one-period debt contract finances investment in both periods, is not straightforward. Since conditional upon success, the firm under an equity contract does not need to borrow in the second period but a debt issuing firm may still need to borrow, the second period investment ought to be closer to the efficient outcome (i.e., the zero NPV cut-off) under the equity contract. On the other hand, since E only holds partial ownership, his incentive to invest productively in the second period is weakened resulting in inefficient liquidation decisions in the second period. Which effect dominates will depend upon the value of α; this in turn will depend upon the amount that has to be raised externally in the first period.

This completes our description of the second period investment decision under a first period equity contract.

Turning now to the first period, assume that E has a cash holding of $c < 1$ and the state of the project is a_1. Suppose $(1 - c)$ is raised through issuing a fraction $(1 - \alpha)$ of equity. For E to invest, both the *IC* and the *IR* constraint must be satisfied. The former is given by

$$\alpha\big[pH_1(\alpha, a_1) + (1 - p)H_0(\alpha)\big] \geqslant B + \alpha H_0(\alpha). \tag{16}$$

For the *IR* constraint, notice that since we assume without loss of generality that if E did not invest today, he would invest in the second period by borrowing, we have

$$\alpha\big[pH_1(\alpha, a_1) + (1 - p)H_0(\alpha)\big] \geqslant P(c). \tag{17}$$

Finally, the outside investors must get break even on the amount $(1 - c)$ that they invest in period 1, i.e.

$$(1 - \alpha)\big[pH_1(\alpha, a_1) + (1 - p)H_0(\alpha)\big] = 1 - c. \tag{18}$$

Let a^c continue to denote the marginal state in period 1 at which E with cash holding c invests.

One can now a prove an analogous result to Proposition 1—the existence of a critical level of $\hat{c}$ such that if $c < \hat{c}$, at the marginal state, the incentive constraint (*IC*) will be binding while the *IR* constraint binds for cash holding in excess of that level. When the moral hazard problem in the second period is sufficiently severe—in particular, if E cannot invest in period 2 with zero cash holding—we can establish the following analogues of Proposition 2 and Proposition 4.[23]

Proposition 5. *Suppose that $P(0) = 0$, then*

(i) *For all $c < \hat{c}$ the marginal state a^c must be decreasing in c.*
(ii) *There exists $c^{**} < B$ such that for all $c \in (c^{**}, B)$, a^c must be increasing in c.*

Proof. Please see Appendix A. □

[23] We have been unable to establish an exact analogue of either Proposition 2 or Proposition 4 for the more general situation.

5.1. Many periods

In this section, we extend our model to allow for more periods. So assume that the firm is active in any period $t \in \{1, 2, \ldots, T\}$. In each period, a state a will be realized and the firm has the option to invest. We assume that in each period, a has an identical distribution $F(\cdot)$ on $[\underline{a}, \bar{a}]$. We simplify analysis by assuming that the risk-free interest is zero in all these periods and that E maximizes the undiscounted cash holding at the end period $T+1$. One way to justify a model with a terminal period is to appeal to the fact that many entrepreneurial firms sell out shortly after going public. If the new buyer is a large firm that can be considered financially unconstrained, then the objective of E is to maximize the value of the firm's terminal period $T+1$ cash accumulated from undertaking projects in the earlier periods.

We maintain our earlier assumption that financing in any period is through use of simple one period debt contracts. Denote by $V^t(x)$ the expected payoff to E at period t, $t \leqslant T$ when his cash holding is x. To obtain $V^t(x)$, observe first that $V^T(x) = P(x)$. Now suppose $V^t(x)$ is defined for all $t = T, T-1, \ldots, (T-m+1)$ and let $t = T-m$. Let x be the cash holding of E at t. If $x > 1$ and given a, if E invests in state a, then his payoff is given by

$$V_I^t(a, x) = pV^{t+1}\big(a + (x-1)\big) + (1-p)V^{t+1}(x-1).$$

If however E does not invest, then

$$V_N^t(a, x) = V^{t+1}(x).$$

Let $V^t(a, x) = \max\{V_I^t(a, x), V_N^t(a, x)\}$ and define

$$V^t(x) = \int_{a \in A} V^t(a, x)\,\mathrm{d}F(a), \quad \text{when } A \equiv [\underline{a}, \bar{a}]$$

Now if $x < 1$ and E wants to invest, he has to raise $(1-x)$. Moreover after raising this amount, E must invest productively. Assuming debt in each period is fairly priced and letting $c' = a - (1-x)/p$, the following *IC* constraint must be satisfied:

$$V_I^t(a, x) = pV^{t+1}\big(c'(0)\big) \geqslant B + V^{t+1}(0). \qquad (IC) \tag{19}$$

Further, E will invest in a only if the *IR* condition is satisfied, i.e., if

$$V_I^t(a, x) \geqslant V^{t+1}(x). \qquad (IR) \tag{20}$$

Defining a_x^t as the infimum of a for which both (*IC*) and (*IR*) constraints are satisfied, we can then define

$$V^t(x) = \int_{a_x^t}^{\bar{a}} V_I^t(a, x)\,\mathrm{d}F(a) + F(a_x^t)V^{t+1}(x).$$

This completes the construction of $V^t(\cdot)$, $t = 1, 2, \ldots, T$.

For any $t < T$, let c^t satisfy the following equation

$$V^{t+1}\big(c^t\big) = B + V^{t+1}(0).$$

It is possible to show that for any t, $V^t(x) - V^t(y) \geqslant (x-y)$ for any x, y with $x > y$. Strict inequality holding if $y < B$. Using now an argument very similar to prove Claim 1, it is now possible to show that for every $t < T$, c^t exists and is unique. Moreover, $c^t < B$.

If a_c^t is the marginal state at which E with cash holding c invests in period t, then one can also obtain an exact analogue of Proposition 1: for every $t < T$, (i) if $c < c^t$, the (IC) constraint binds at a_c^t and the (IR) constraint does not, while (ii) if $c > c^t$, the (IR) constraint binds at a_c^t and the (IC) constraint does not.

The result that is most robust to the extension to this multi-period setting is the analogue of Proposition 3, i.e. the fact that the slope of the marginal state $a_c^t(c)$, for any t, changes discontinuously at c^t.

Unfortunately, in this most general set up, we have been unable to prove an analogue of Proposition 4. However, Fig. 3 shows that, for a particular example with $T = 4$, non-monotonicity is exhibited in every period except the last.[24] Further, the non-monotonic response of investment to increases in cash balance for every $t \in T$ does hold for some variants of the model outlined above. In an earlier version of the paper, two such variants were considered.

In the first variant, it was assumed that if at any t, E invests and fails, then he cannot invest any more next period onwards. On the other hand, if the investment at t is successful or if E did not invest at t, then E remains active and can invest in future periods. In the second variant, we considered the polar opposite situation where the firm was assumed to be sold immediately to a financially unconstrained buyer if at any period, E's investment was successful. If however E's investment failed or E did not invest in some period t, he could continue investing in future periods.

In both of these variants, we are able to show that for an active firm in time $t < T$, there exists a range of cash holdings over which E's investment is decreasing in his cash holding.[25]

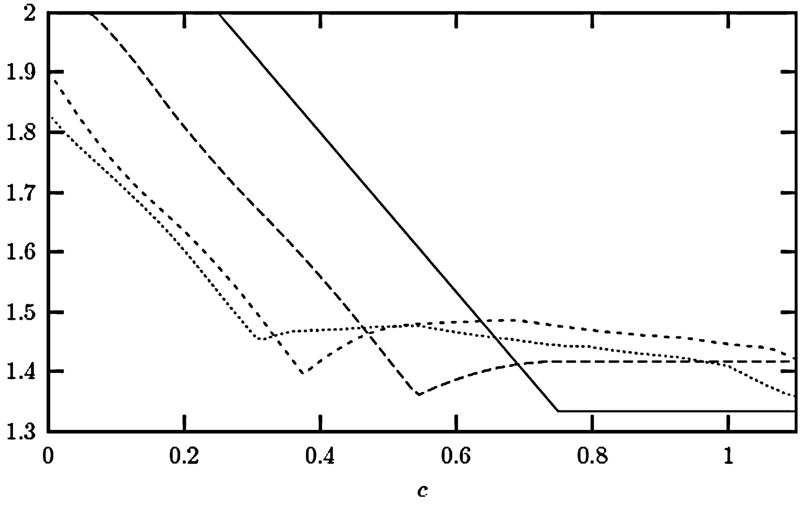

Fig. 3. Non-monotonicity of hurdle rates as functions of the initial cash balance c: Solid line, a_c^1; dashed line, a_c^2; short-dashed line, a_c^3; dotted line, a_c^4, $p = 3/4$, $\bar{a} = 2$, $B = 3/4$.

[24] For Fig. 3, $F(a)$ is uniform over $[\frac{1}{p}, 2]$, $p = 0.75$ and $B = 0.75$.

[25] The proof in both these cases consist of first showing that $\frac{\mathrm{d}V^t(x)}{\mathrm{d}x} = 1$ for $x = B$ for any $t \leqslant T$, and using an argument very similar to the one provided for the proof of Proposition 4. The proof is available from the authors on request.

6. Conclusion

It is generally assumed that financially constrained firms invest more when their net worth increases or when interest rates are lowered. We argue that while single period models do support these presumptions, the results do not extend to multi-period settings. A straightforward extension of Holmstrom and Tirole's (1997, 1998) model to a multi-period setting shows that firms' investment can be non-monotonic in the level of liquid balances. Further, a downward shift in the yield curve will also reduce investment in the current period: interest rate cuts stimulate investment only if the yield curve also becomes more positively sloped.

Our results have several implications for monetary policy. First, they indicate that the effect of monetary policy aimed at stimulating firm investment through the balance sheet channel may be more muted than usually supposed, especially if firms already have accumulated significant liquid balances. The fact that firms tend to accumulate cash balances in recessions (Almeida et al., 2002), combined with our result that firms may become conservative and cut back on investment when they have more cash balance, suggests that there may be an additional channel through which aggregate demand is affected during downturns. This channel may be especially relevant in the context of the Japanese experience, where firms have steadily increased their holdings of liquid balances through the nineties.

Our result that interest rate policy may fail to stimulate investment unless the yield curve also becomes steeper is consistent with the evidence in Estrella and Mishkin (1998) that an inverted yield curve foreshadows recessions. The Japanese experience is also relevant here: while nominal interest rates in Japan have been close to zero, the central bank has sharply increased the base money supply, thereby fostering the expectation of high future inflation. Consequently, even though the nominal yield curve is upward sloping, real interest rates are likely to be lower in the long term than in the short term. Under these conditions, our model implies that firms will postpone investment.

Finally, our model addresses and reconciles a number of stylized empirical facts with theory, and also generates some new empirical implications. First, it provides a rigorous framework to explain why cash is more valuable for financially constrained firms, and why the marginal value of cash is decreasing in the level of cash. Faulkender and Wang (2006) find support for precisely these implications (specifically, these constitute their Hypothesis 1 and Hypothesis 3, respectively).[26] Second, we show in a rigorous framework that the magnitude of the response of investment to cash flow shocks (or shocks to liquidity) is non-monotonic, consistent with empirical evidence in Kaplan and Zingales (1997) and Cleary (1999), among others. Third, our model predicts that firms that have very low or very high levels of cash balances will pick up their capital expenditures faster at the end of a recession than those with intermediate levels of cash holdings, ceteris paribus. Finally, our model implies that economic activity will be positively related to future interest rates but negatively related to current interest rates, and the effect of the

[26] The marginal value of cash is the derivative of the expression on the left-hand side of the *IC* or *IR* constraints, and is $P'(a_1 - D) \geqslant 1$ if the firm invests in the first period, and $P'(c) > 1$ for $c < B$ if the firm does not invest in the first period. For $c < B$, the expected value of this expression taken over the set of possible states in the first period will be strictly greater than unity. On the other hand, for a financially unconstrained firm ($c > B$), the expected value is unity. While we require no assumptions regarding the second derivative of the $P(\cdot)$ function for our results, concavity is a reasonable assumption—as noted earlier, it is consistent with a hedging motive and holds, in particular, if $F(\cdot)$ is uniform. Concavity of $P(\cdot)$ implies that the marginal value of c is decreasing for $c < B$.

latter will be stronger in magnitude than the former. To the best of our knowledge, these last two implications have not been investigated.

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Appendix A

Proof of Claim 1. At $c = 0$, $P(0) < P(0) + B$. Further, for $c = B$, it follows from Eq. (2) that E can invest in all productive states in period 2, i.e., $a_B = a^*$. Hence, $P(B) = B + \int_{a^*}^{\bar{a}} (pa - 1)\,\mathrm{d}F$. On the other hand, with zero cash holding, E will not be able to invest in all productive states (i.e., $a_0 > a^*$). Thus, it follows that $P(B) > B + \int_{a_0}^{\bar{a}} (pa - 1)\,\mathrm{d}F = B + P(0)$. By continuity of $P(\cdot)$, it then follows that there exists a c^0 with $c^0 < B$ such that $P(c^0) = B + P(0)$. Since $P(\cdot)$ is an increasing function of c, c^0 is unique. □

Proof of Proposition 1. To prove Proposition 1, we need the following lemma.

Lemma 1. *For all $c < c^0$, in any state in which E invests in period 1 (including the marginal state), his payoff is strictly greater than $P(c)$.*

Proof. Because of the *IC* constraint, in any state a that E invests productively, his expected payoff must be at least $B + P(0)$. By not investing in that state, E can get at most $P(c)$. Since for $c < c^0$, $P(c) < P(c^0) = B + P(0)$, the result follows. □

We now return to the proof of the proposition.

(i) Consider $c < c^0$. By Lemma 1, at a^c, the *IR* constraint does not bind, and thus by continuity, there exists states slightly lower than a^c where E will prefer to invest today rather than not invest. If *IC* constraint did not bind at the marginal state a^c, *IC* constraint will also be satisfied for states lower than a^c but sufficiently close to it. This however contradicts the definition of a^c. Thus, *IC* constraint must bind for all $c < c^0$ at the marginal state a^c.

(ii) Consider $c > c^0$ and assume that *IR* constraint does not bind at the marginal state. Now if the *IC* constraint also does not bind at a^c, by an argument analogous to the earlier part, we can show that E is better off investing in a state slightly lower than a^c. Therefore, the *IC* constraint must bind at a^c if *IR* does not bind. This means that $p[P(a^c - D) - P(0)] = B$ where $D = (1 - c)/p$. Further, since *IR* does not bind, we have $p[P(a^c - D) - P(0)] + P(0) > P(c)$.We then have $B + P(0) = P(c^0) > P(c)$ but this implies that $c < c^0$, a contradiction to our hypothesis that $c > c^0$. This contradiction establishes that the *IR* constraint must bind for $c > c^0$, i.e., $P(c) = p[P(a^c - D) - P(0)] + P(0)$. Since $c > c^0$, we have $P(c) > P(c^0) = B + P(0)$ and thus $p[P(a^c - D) - P(0)] + P(0) > B + P(0)$. Thus, the *IC* constraint does not bind at $c > c^0$. □

Proof of Proposition 3. For $c < c^0$, in the marginal state a^c, the *IR* constraint is not binding but the *IC* constraint is. It follows from differentiating the left-hand side of the binding *IC* constraint that $\frac{\mathrm{d}a^c}{\mathrm{d}c} = -\frac{1}{p}$.

On the other hand, for $c > c^0$, *IR* constraint must bind at the marginal state. Thus, $P(c) = pP(y(c)) + (1-p)P(0)$ and $y(c) = a^c - \frac{(1-c)}{p}$. Totally differentiating, we have

$$P'(c) = pP'\big(y(c)\big)\left[\frac{\mathrm{d}a^c}{\mathrm{d}c} + \frac{1}{p}\right].$$

Simplifying, one obtains

$$\frac{\mathrm{d}a^c}{\mathrm{d}c} = -\frac{1}{p} + \frac{P'(c)}{pP'(y(c))}.$$

Hence, the result follows. □

Proof of Proposition 4′. From Proposition 1, we know that at $c = c^0$, the *IC* constraint binds and thus

$$p\big[P\big(y\big(c^0\big)\big) - P(0)\big] = B.$$

Now

$$P\big(y\big(c^0\big)\big) - P(0) = \int_{a_{y(c^0)}}^{a_0} [pa_2 - 1]\,\mathrm{d}F + y\big(c^0\big).$$

Since at a_0, we have $pa_0 - 1 = B$, for all $a_2 \in [a_{y(c^0)}, a_0]$, we have $pa_2 - 1 < B$. Thus

$$B = p\big[P\big(y\big(c^0\big)\big) - P(0)\big] < p\left[\int_{a_{y(c^0)}}^{a_0} B\,\mathrm{d}F + y\big(c^0\big)\right] < p\big[B + y\big(c^0\big)\big].$$

Since $p \leqslant 1/2$, we then have $B < y(c^0)$. Thus for all $c \geqslant c^0$, we have $y(c) > B$. The result now follows from arguments used in the proof of Proposition 4. □

Proof of Proposition 5. (i) Since $0 \leqslant H_0(\alpha) \leqslant P(0) = 0$, we have $H_0(\alpha) = 0$. Thus, for all $c < \hat{c}$ where the *IC* constraint binds at the marginal state a^c, we have, from the *IC* constraint and the investors break even condition (Eqs. (16) and (18))

$$pH_1\big(\alpha, a^c\big) - (1-c) = B \tag{A.1}$$

where $H_1(\alpha, a^c) = a^c + \int_{a^\alpha}^{\bar{a}} (pa - 1)\,\mathrm{d}F$. Totally differentiating the above equation with respect to c, we get

$$\frac{\mathrm{d}a^c}{\mathrm{d}c} - \big[pa^\alpha - 1\big]\frac{\mathrm{d}\alpha}{\mathrm{d}c}\frac{\mathrm{d}a^\alpha}{\mathrm{d}\alpha} + 1 = 0.$$

Clearly $\frac{\mathrm{d}a^\alpha}{\mathrm{d}\alpha} < 0$. We now check that $\frac{\mathrm{d}\alpha}{\mathrm{d}c} > 0$. Use (A.1) to get $p\,\mathrm{d}H_1 = -\mathrm{d}c$. Substitute in (18) to get $(1-\alpha)(-\mathrm{d}c) - \mathrm{d}\alpha\, pH_1 = -\mathrm{d}c$. Hence $\alpha\,\mathrm{d}c = \mathrm{d}\alpha\, pH_1$ or $\frac{\mathrm{d}\alpha}{\mathrm{d}c} > 0$. Thus,

$$\frac{\mathrm{d}a^c}{\mathrm{d}c} = -1 + \left[pa^\alpha - 1\right]\frac{\mathrm{d}\alpha}{\mathrm{d}c}\frac{\mathrm{d}a^\alpha}{\mathrm{d}\alpha} < 0$$

and the result is proved.

(ii) The expected payoff to E by investing at a^c equals $\alpha p H_1(\alpha, a^c)$ and the payoff to the investors is $(1-\alpha)pH_1(\alpha, a^c) = 1 - c$. Since E's payoff must be at least $P(c)$ which in turn is strictly greater than c, it follows that $pH_1(\alpha, a^c) > 1$. Thus $\alpha > c$ and for c close to B, we have $B/\alpha < 1$. This implies $a^\alpha = a^*$. Thus, for all such c, we have

$$H_1\big(\alpha, a^c\big) = a^c + \int_{a^*}^{\bar{a}} (pa - 1)\,\mathrm{d}F.$$

Now for c close to B, the *IR* constraint must be binding. Rewriting the *IR* constraint, we then get

$$p\left[a^c + \int_{a^*}^{\bar{a}} (pa - 1)\,\mathrm{d}F\right] - (1 - c) = P(c)$$

where $P(c) = c + \int_{a_c}^{\bar{a}} [pa - 1]\,\mathrm{d}F(\cdot)$. Totally differentiating both sides with respect to c, we then get

$$p\frac{\mathrm{d}a^c}{\mathrm{d}c} + 1 = P'(c).$$

Since $c < B$, $P'(c) > 1$ (see the proof of Proposition 4), we then have $\frac{\mathrm{d}a^c}{\mathrm{d}c} > 0$ and the result is proved. □

Appendix B. The effect of more uncertainty in the second period

To illustrate how the impact of greater uncertainty can be the opposite of that in the real options framework, we consider a case where there are two equally likely states possible in the second period, denoted by a^1 and a^2, where $a^1 > a^2$.

Let the corresponding minimal amounts of cash needed to finance these projects be $x^1 = B - (pa^1 - 1)$ and $x^2 = B - (pa^2 - 1)$.

Clearly, $x^1 < x^2$. We assume that $x^1 > 0$, so that $P(0) = 0$.

In this case, $x^2 > c^0 > x^1$, since $P(x^1) = B - (pa^1 - 1) + (1/2)(pa^1 - 1) < B$ and $P(x^2) = B - (pa_2^2 - 1) + (1/2)(pa^1 - 1) + (1/2)(pa^2 - 1) > B$. Consider $c = x^2$. For this c, the *IR* constraint must bind at the marginal state a^c and we thus have $pP(a^c - \frac{1-c}{p}) = P(c)$. Notice that $a^c - \frac{1-c}{p} > c = x^2$, so that if success occurs, E is able to invest next period in either project. Hence the *IR* constraint is $pa^c - 1 = (1-p)[p(1/2)(a^1 + a^2) - 1]$.

Consider now a mean preserving spread of the distribution in the second period: let a^1 increase to $a^{1'}$ and a^2 decrease to $a^{2'}$ keeping the expected value unchanged. Clearly, $x^{2'} > x^2$ and $x^{1'} < x^1$. At $c = x^2$, it is no longer possible to invest in the less productive project in the second period. Depending on whether or not it is still possible to invest in the less productive project when success occurs from investing at the marginal state at $t = 1$, the *IR* constraint at $c = x^2$ is either $pa^{c'} - 1 = (1/2)(1-p)(pa^{1'} - 1) - (1/2)(pa^{2'} - 1)$ or $pa^{c'} - 1(1/2)(1-p)(pa^{1'} - 1)$, respectively. Since $(1/2)p(a^1 + a^2) - 1(1/2)p(a^{1'} + a^{2'}) - 1 > (1/2)(pa^{1'} - 1)$, it follows that

$a^{c'} < a^c$, i.e. the firm is more likely to invest. Since the firm's payoff at c is $\int_{a^c}^{\bar{a}} \mathrm{Max}[pP(a^c - \frac{1-c}{p}), P(c)]\,\mathrm{d}F$ and $P(c)$ is lower, it follows immediately that firm value also falls.[27]

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[27] For $c < c^0$, the *IC* constraint binds. In this case, a mean preserving spread either leaves the marginal project unchanged, with no effect on firm value, or *raises* the marginal project, if the spread is sufficiently large, so that even if the period one project is successful, E is unable to invest in the lower productivity project in the second period. In the latter case, firm value falls.

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